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Some properties of rapidly varying sequences

D. Djurčić^{a,1}, Lj.D.R. Kočinac^{b,*,1}, M.R. Žižović^{a,1}

^a Technical Faculty, University of Kragujevac, Svetog Save 65, 32000 Čačak, Serbia and Montenegro

^b Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia and Montenegro

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Abstract

We investigate the class $R_{\infty,s}$ of rapidly varying sequences from different points of view. It is shown: (1) a sequence $(c_n)_{n \in \mathbb{N}}$ of positive real numbers is rapidly varying if and only if the corresponding function $x \rightarrow c_{[x]}$, $x \geq 1$, is rapidly varying; (2) the class $R_{\infty,s}$ satisfies some selection properties, as well as game-theoretical and Ramsey-theoretical conditions.

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1. Introduction

A function $f : [a, +\infty) \rightarrow (0, +\infty)$, $a > 0$, is said to be *slowly varying* in the sense of Karamata [1] if it is measurable and satisfies the following asymptotic condition:

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = 1, \quad \lambda > 0. \quad (1)$$

The class of slowly varying functions we shall denote by SV_f .

* Corresponding author. Fax: +381 18 533 014.

E-mail addresses: dragandj@tfc.kg.ac.yu (D. Djurčić), lkocinac@ptt.yu (Lj.D.R. Kočinac), zizo@tfc.kg.ac.yu (M.R. Žižović).

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A sequence $(c_n)_{n \in \mathbb{N}}$ of positive real numbers is said to be *slowly varying* in the sense of Karamata [2] if the following asymptotic condition is satisfied:

$$\lim_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = 1, \quad \lambda > 0. \quad (2)$$

The class of slowly varying sequences we denote by \mathbf{SV}_s .

Slow variability in the sense of Karamata is an important asymptotic property in analysis of divergent processes [1].

In [2], R. Bojanić and E. Seneta gave (see also [6]) a qualitative relation between sequential property (2) and functional property (1) and established a unique concept of interpretation and development of the theory of slow variability in the sense of Karamata.

Theorem BS. *For a sequence $(c_n)_{n \in \mathbb{N}}$ of positive real numbers the following are equivalent:*

- (a) $(c_n)_{n \in \mathbb{N}}$ belongs to the class \mathbf{SV}_s ;
- (b) the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, is in the class \mathbf{SV}_f .

The corresponding results based on Theorem BS which treat O -regular variability, extended regular variability and SO -regular variability can be found in the papers [3–5].

M. Tasković has shown in the paper [12] an important generalization of Theorem BS for translational slow variability.

A function $f: [a, +\infty) \rightarrow (0, +\infty)$, $a > 0$, is said to be *rapidly varying* in the sense of de Haan [7] if it is measurable and satisfies the asymptotic condition

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = 0, \quad 0 < \lambda < 1. \quad (3)$$

The class of rapidly varying functions we shall denote by $\mathbf{R}_{\infty, f}$.

A sequence $(c_n)_{n \in \mathbb{N}}$ of positive real numbers is said to be *rapidly varying* if the following asymptotic condition is satisfied:

$$\lim_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = 0, \quad 0 < \lambda < 1. \quad (4)$$

The class of rapidly varying sequences we denote by $\mathbf{R}_{\infty, s}$.

Rapid variability given in the sequential form (4) and functional form (3) is rapid variability in the sense of de Haan of index $+\infty$, and in the case of monotone and unbounded mappings it is conjugate with slow variability in the sense of Karamata by the generalized inverse (see [1]).

Rapid variability given by (3) and (4), as a dual of asymptotic property of slow variability given by (1) and (2), is an important object of asymptotic analysis (see [1]).

2. Rapidly varying functions and sequences

The following theorem is a Bojanić–Seneta type assertion for rapid variability given by (3) and (4).

Theorem 2.1. *For a sequence $(c_n)_{n \in \mathbb{N}}$ of positive real numbers the following are equivalent:*

- (a) $(c_n)_{n \in \mathbb{N}}$ belongs to the class $\mathbf{R}_{\infty, s}$;
- (b) the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, is in the class $\mathbf{R}_{\infty, f}$.

Proof. (a) \Rightarrow (b): Let $\lambda \in (0, 1)$. Then for every $\alpha \in (\lambda, 1)$ we have $\lim_{n \rightarrow +\infty} \frac{c_{[\alpha n]}}{c_n} = 0$. Let us show that for a given $\varepsilon > 0$ there exist an interval $[A, B]$ which is a proper subset of $(\lambda, 1)$ and $n_0 \in \mathbb{N}$ such that $\frac{c_{[\alpha n]}}{c_n} < \varepsilon$ for each $n \geq n_0$ and each $\alpha \in [A, B]$. For an arbitrary and fixed $\alpha \in (\lambda, 1)$ define $n_\alpha \in \mathbb{N}$ in the following way:

$$n_\alpha = \begin{cases} 1, & \text{if } \frac{c_{[\alpha n]}}{c_n} < \varepsilon \text{ for each } n \in \mathbb{N}; \\ 1 + \max\{n \in \mathbb{N}: \frac{c_{[\alpha n]}}{c_n} \geq \varepsilon\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $1 \leq n_\alpha < +\infty$ for each considered α . For each $k \in \mathbb{N}$ define

$$A_k = \{\alpha \in (\lambda, 1): n_\alpha > k\}.$$

Then $(A_k)_{k \in \mathbb{N}}$ is a nonincreasing sequence of sets such that $\bigcap_{k \in \mathbb{N}} A_k = \emptyset$. We prove that not all sets A_k are dense in $(\lambda, 1)$. If $k \in \mathbb{N}$ is fixed and $\alpha \in A_k$, then

$$\frac{c_{[\alpha(n_\alpha-1)]}}{c_{n_\alpha-1}} \geq \varepsilon$$

and there is some $\delta_\alpha > 0$ such that for each $t \in [\alpha, \alpha + \delta_\alpha) \subset (\lambda, 1)$ we have

$$\frac{c_{[t(n_\alpha-1)]}}{c_{n_\alpha-1}} = \frac{c_{[\alpha(n_\alpha-1)]}}{c_{n_\alpha-1}} \geq \varepsilon.$$

This means that each $t \in (\alpha, \alpha + \delta_\alpha)$ belongs to the set A_k , since $n_t \geq (n_\alpha - 1) + 1 > k$. It follows that if $\alpha \in A_k$, then $(\alpha, \alpha + \delta_\alpha) \subset A_k$. If we assume that some of the sets A_k is dense in $(\lambda, 1)$, then the set $\text{Int}(A_k)$ is also dense in $(\lambda, 1)$. If, on the other side, we suppose that all the sets A_k are dense in $(\lambda, 1)$, then $(\text{Int}(A_k))_{k \in \mathbb{N}}$ is a sequence of open, dense subsets of the set $(\lambda, 1)$ of the second category. It follows that the set $\bigcap_{k \in \mathbb{N}} A_k$ is dense in $(\lambda, 1)$ and thus nonempty, and we have a contradiction. Therefore, there is $n_0 \in \mathbb{N}$ such that the set A_{n_0} is not dense in $(\lambda, 1)$. Consequently, there is an interval $[A, B]$, a proper subset of $(\lambda, 1)$, such that

$$[A, B] \subset (\lambda, 1) \setminus A_{n_0} = \{\alpha \in (\lambda, 1): n_\alpha \leq n_0\}.$$

From here it follows that $n_\alpha \leq n_0$ for each $\alpha \in [A, B]$, and thus for each $n \geq n_0 \geq n_\alpha$ and each $\alpha \in [A, B]$ it holds $\frac{c_{[\alpha n]}}{c_n} < \varepsilon$.

We conclude that for $\lambda \in (0, 1)$ and each $x \in [1, +\infty)$ large enough, we have

$$\frac{c_{[\lambda x]}}{c_{[x]}} = \frac{c_{[t[\eta[x]]]}}{c_{[\eta[x]]}} \cdot \frac{c_{[\eta[x]]}}{c_{[x]}},$$

where $t = t(x) \in [A, B]$ and $\eta = \frac{2\lambda}{A+B}$.

Since $\eta \in (0, 1)$ we have

$$\limsup_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[x]}} \leq \varepsilon \cdot \limsup_{x \rightarrow +\infty} \frac{c_{[\eta[x]]}}{c_{[x]}} = 0.$$

This means that the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, belongs to the class $R_{\infty, f}$.

(b) \Rightarrow (a): It is trivial, because for an arbitrary and fixed $\lambda \in (0, 1)$ we have

$$\lim_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = \lim_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[x]}} = 0. \quad \square$$

This theorem allows a unique and unified development of the theory of rapidly varying sequences and rapidly varying functions given by (4) and (3), in the same way as Theorem BS allowed that in the theory of slowly varying sequences and functions (see [2]).

Corollary 2.2. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Then $(c_n)_{n \in \mathbb{N}}$ belongs to the class $R_{\infty, s}$ if and only if

$$\lim_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = \infty, \quad \lambda > 1. \quad (5)$$

Proof. Let $(c_n)_{n \in \mathbb{N}}$ belongs to the class $R_{\infty, s}$. By Theorem 2.1 the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, belongs to the class $R_{\infty, f}$. Therefore, for $\lambda > 1$ it holds

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = \lim_{x \rightarrow +\infty} \frac{1}{\frac{f(x)}{f(\lambda x)}} = \frac{1}{\lim_{x \rightarrow +\infty} \frac{f(\frac{1}{\lambda} x)}{f(x)}} = +\infty. \quad (6)$$

In other words, for the considered (arbitrary and fixed) $\lambda > 1$, it holds $\lim_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty$, i.e., (5).

Conversely, suppose (5) is satisfied. Fix $\lambda > 1$. Then for each $\alpha \in (1, \lambda)$ we have $\lim_{n \rightarrow +\infty} \frac{c_{[\alpha n]}}{c_n} = +\infty$. For any $\varepsilon > 0$ and any $\alpha \in (1, \lambda)$ define $n_\alpha \in \mathbb{N}$ as follows:

$$n_\alpha = \begin{cases} 1, & \text{if } \frac{c_{[\alpha n]}}{c_n} > \varepsilon \text{ for each } n \in \mathbb{N}; \\ 1 + \max\{n \in \mathbb{N} : \frac{c_{[\alpha n]}}{c_n} \leq \varepsilon\}, & \text{otherwise.} \end{cases}$$

Clearly, for the considered α , $1 \leq n_\alpha < +\infty$. For each positive integer k define

$$A_k = \{\alpha \in (1, \lambda) : n_\alpha > k\}.$$

Similarly to the proof of (a) \Rightarrow (b) of Theorem 2.1 we conclude that there is $n_0 \in \mathbb{N}$ such that the set A_{n_0} is not dense in $(1, \lambda)$. So, there is an interval $[A, B]$, a proper subset of $(1, \lambda)$, such that

$$[A, B] \subset (1, \lambda) \setminus A_{n_0} = \{\alpha \in (1, \lambda) : n_\alpha \leq n_0\},$$

i.e., $n_\alpha \leq n_0$ for each $\alpha \in [A, B]$. Thus for each $\alpha \in [A, B]$ and each $n \geq n_0 \geq n_\alpha$ one has $\frac{c_{[\alpha n]}}{c_n} > \varepsilon$. From here we derive that for $\lambda \in (1, +\infty)$ and every $x \geq 1$ sufficiently large, we have

$$\frac{c_{[\lambda x]}}{c_{[x]}} = \frac{c_{[t[\eta[x]]]}}{c_{[\eta[x]]}} \cdot \frac{c_{[\eta[x]]}}{c_{[x]}},$$

where $t = t(x) \in [A, B]$ and $\eta = \frac{2\lambda}{A+B}$.

As $\eta > 1$ we have

$$\liminf_{x \rightarrow +\infty} \frac{c_{[\lambda x]}}{c_{[x]}} \geq \varepsilon \cdot \liminf_{x \rightarrow +\infty} \frac{c_{[\eta[x]]}}{c_{[x]}} = +\infty,$$

which means that the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, satisfies $\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = +\infty$ for any $\lambda > 1$. For any $\lambda \in (0, 1)$ we then have

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = \lim_{x \rightarrow +\infty} \frac{1}{\frac{f(x)}{f(\lambda x)}} = \frac{1}{\lim_{x \rightarrow +\infty} \frac{f(\frac{1}{\lambda} x)}{f(x)}} = 0,$$

i.e., $f \in R_{\infty, f}$. Again by Theorem 2.1 we conclude $(c_n)_{n \in \mathbb{N}} \in R_{\infty, s}$. \square

Corollary 2.3. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers belonging to the class $R_{\infty, s}$. Then the limit in (4) is uniform for $\lambda \in (0, A]$, $A \in (0, 1)$, and the limit in (5) is uniform for $\lambda \in [B, \infty)$, $B > 1$.

Proof. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers satisfying (4). Then the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, by Theorem 2.1, belongs to $\mathbf{R}_{\infty, f}$. By [1], for any $A \in (0, 1)$,

$$\limsup_{x \rightarrow +\infty, \lambda \in (0, A]} \frac{c_{[\lambda x]}}{c_{[x]}} = 0.$$

Because for every $\lambda \in (0, A]$ and $t \geq t_0(\lambda)$ sufficiently large it holds

$$\sup_{n \geq [t]+1} \frac{c_{[\lambda n]}}{c_n} \leq \sup_{x \geq t} \frac{c_{[\lambda x]}}{c_{[x]}},$$

we have

$$\sup_{\lambda \in (0, A]} \sup_{n \geq [t]+1} \frac{c_{[\lambda n]}}{c_n} \leq \sup_{\lambda \in (0, A]} \sup_{x \geq t} \frac{c_{[\lambda x]}}{c_{[x]}},$$

and thus

$$\lim_{t \rightarrow +\infty} \sup_{n \geq [t]+1} \sup_{\lambda \in (0, A]} \frac{c_{[\lambda n]}}{c_n} \leq \lim_{t \rightarrow +\infty} \sup_{x \geq t} \sup_{\lambda \in (0, A]} \frac{c_{[\lambda x]}}{c_{[x]}}.$$

So we have

$$\limsup_{n \rightarrow +\infty} \sup_{\lambda \in (0, A]} \frac{c_{[\lambda n]}}{c_n} \leq \limsup_{x \rightarrow +\infty} \sup_{\lambda \in (0, A]} \frac{c_{[\lambda x]}}{c_{[x]}} = 0.$$

Therefore,

$$\lim_{n \rightarrow +\infty} \sup_{\lambda \in (0, A]} \frac{c_{[\lambda n]}}{c_n} = 0,$$

so that the limit in (4) is uniform for $\lambda \in (0, A]$.

Let now $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers which satisfies (5). Then by Corollary 2.2, $(c_n)_{n \in \mathbb{N}} \in \mathbf{R}_{\infty, s}$ and thus the function f defined by $f(x) = c_{[x]}$, $x \geq 1$, by Theorem 2.1, belongs to $\mathbf{R}_{\infty, f}$. By [1] and (6) we have that for any $B > 1$,

$$\liminf_{x \rightarrow +\infty, \lambda \in [B, +\infty)} \frac{c_{[\lambda x]}}{c_{[x]}} = +\infty.$$

Since for every $\lambda \in [B, +\infty)$ and $t \geq t_0(\lambda)$ large enough it holds

$$\inf_{n \geq [t]+1} \frac{c_{[\lambda n]}}{c_n} \geq \inf_{x \geq t} \frac{c_{[\lambda x]}}{c_{[x]}},$$

one obtains

$$\inf_{\lambda \in [B, +\infty)} \inf_{n \geq [t]+1} \frac{c_{[\lambda n]}}{c_n} \geq \inf_{\lambda \in [B, +\infty)} \inf_{x \geq t} \frac{c_{[\lambda x]}}{c_{[x]}}.$$

In this way

$$\lim_{t \rightarrow +\infty} \inf_{n \geq [t]+1} \inf_{\lambda \in [B, +\infty)} \frac{c_{[\lambda n]}}{c_n} \geq \lim_{t \rightarrow +\infty} \inf_{x \geq t} \inf_{\lambda \in [B, +\infty)} \frac{c_{[\lambda x]}}{c_{[x]}},$$

and we get

$$\liminf_{n \rightarrow +\infty} \inf_{\lambda \in [B, +\infty)} \frac{c_{[\lambda n]}}{c_n} \geq \liminf_{x \rightarrow +\infty} \inf_{\lambda \in [B, +\infty)} \frac{c_{[\lambda x]}}{c_{[x]}} = +\infty$$

and

$$\lim_{n \rightarrow +\infty} \inf_{\lambda \in [B, +\infty)} \frac{c_{[\lambda n]}}{c_n} = +\infty.$$

This shows that the limit in (5) is uniform for $\lambda \in [B, +\infty)$. \square

Corollary 2.4. *If a sequence $(c_n)_{n \in \mathbb{N}}$ belongs to $\mathbf{R}_{\infty, s}$, then $\lim_{n \rightarrow \infty} c_n = +\infty$.*

Proof. Let $(c_n)_{n \in \mathbb{N}} \in \mathbf{R}_{\infty, s}$. Then by Corollary 2.3, $\lim_{n \rightarrow +\infty} \inf_{\lambda \in [2, +\infty)} \frac{c_{[\lambda n]}}{c_n} = +\infty$. It follows that for $M > 1$ there is $n_0 \in \mathbb{N}$ such that for each $\lambda \in [2, +\infty)$ and each $n \geq n_0$ it holds $\frac{c_{[\lambda n]}}{c_n} > M$. Therefore, for the same λ we have $c_{[\lambda n_0]} > M \cdot c_{n_0}$, and thus for every $n \geq 2n_0$, $c_n > M \cdot c_{n_0}$.

Also, for every $\lambda \in [2, +\infty)$ it holds $\frac{c_{[\lambda \cdot 2n_0]}}{c_{2n_0}} > M$. This means that for the same λ , $c_{[\lambda \cdot 2n_0]} > M \cdot c_{2n_0} > M^2 \cdot c_{n_0}$, which implies that for every $n \geq 4n_0$ we have $c_n > M^2 \cdot c_{n_0}$. Because $M > 1$ and $c_{n_0} > 0$, one concludes that following this procedure for each fixed $\alpha > 0$ one can find some $n_1 = n_1(\alpha)$ in \mathbb{N} with $c_n > \alpha$ for each $n \geq n_1$. So, $\lim_{n \rightarrow +\infty} c_n = +\infty$. \square

Corollary 2.5. *Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Then the following are equivalent:*

- (a) $(c_n)_{n \in \mathbb{N}}$ belongs to the class $\mathbf{R}_{\infty, s}$;
- (b) if $k_i : \mathbb{N} \rightarrow \mathbb{N}$, $i \in \{1, 2\}$, are functions satisfying $\lim_{n \rightarrow +\infty} k_i(n) = +\infty$ and $k_1(n) \leq K \cdot k_2(n)$ for some $K \in (0, 1)$ and $n \geq n_0(K)$, then $c_{k_1(n)} = o(c_{k_2(n)})$ as $n \rightarrow +\infty$.

Proof. (a) \Rightarrow (b): Let $(c_n)_{n \in \mathbb{N}} \in \mathbf{R}_{\infty, s}$ and let the assumptions of (b) be satisfied for functions k_1 and k_2 . Then, using Corollary 2.3, we have

$$\lim_{n \rightarrow +\infty} \frac{c_{k_1(n)}}{c_{k_2(n)}} = \lim_{n \rightarrow +\infty} \frac{\frac{c_{k_1(n)}}{k_2(n)} \cdot k_2(n)}{c_{k_2(n)}} \leq \lim_{n \rightarrow +\infty} \sup_{\lambda \in (0, K]} \frac{c_{[\lambda k_2(n)]}}{c_{k_2(n)}} = 0.$$

(b) \Rightarrow (a): Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers and let $K \in (0, 1)$ be arbitrary and fixed. Also, let $k_1(n) = [Kn]$ and $k_2(n) = n$, $n \in \mathbb{N}$. Since the functions k_1 and k_2 satisfy the conditions in (b), we have

$$\lim_{n \rightarrow +\infty} \frac{c_{[Kn]}}{c_n} = \lim_{n \rightarrow +\infty} \frac{c_{k_1(n)}}{c_{k_2(n)}} = 0,$$

which shows that $(c_n)_{n \in \mathbb{N}}$ belongs to $\mathbf{R}_{\infty, s}$. \square

Remark. A sequence $(b_n)_{n \in \mathbb{N}}$ of positive real numbers is said to be *O-regularly varying* (see [3]) if for every $\lambda > 0$ it is satisfied

$$\bar{k}_b(\lambda) := \limsup_{n \rightarrow +\infty} \frac{b_{[\lambda n]}}{b_n} < +\infty.$$

Observe (see [1,3]) that for any *O-regularly varying* sequence $(b_n)_{n \in \mathbb{N}}$ we have

$$\underline{k}_b(\lambda) := \liminf_{n \rightarrow +\infty} \frac{b_{[\lambda n]}}{b_n} > 0$$

for every $\lambda > 0$.

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence from the class $\mathbf{R}_{\infty, s}$ and let $(b_n)_{n \in \mathbb{N}}$ be an *O-regularly varying* sequence. Consider the sequence $(d_n)_{n \in \mathbb{N}}$ defined by $d_n = \frac{c_n}{b_n}$, $n \in \mathbb{N}$. For every $\lambda > 1$ we have

$$\liminf_{n \rightarrow +\infty} \frac{d_{[\lambda n]}}{d_n} \geq \liminf_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} \cdot \liminf_{n \rightarrow +\infty} \frac{b_n}{b_{[\lambda n]}}.$$

Since, by Corollary 2.2, $\liminf_{n \rightarrow +\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty$, and

$$\liminf_{n \rightarrow +\infty} \frac{b_n}{b_{[\lambda n]}} \geq \liminf_{n \rightarrow +\infty} \inf_{\alpha \in [\frac{1}{\lambda}, \delta]} \frac{b_{[\alpha n]}}{b_n} > 0, \quad \frac{1}{\lambda} < \delta < +\infty,$$

one obtains

$$\lim_{n \rightarrow +\infty} \frac{d_{[\lambda n]}}{d_n} = +\infty.$$

Corollary 2.2 witnesses now that the sequence $(d_n)_{n \in \mathbb{N}}$ belongs to $R_{\infty, s}$, and Corollary 2.4 implies that $\lim_{n \rightarrow +\infty} d_n = +\infty$.

Example 2.6. If $(c_n)_{n \in \mathbb{N}} \in R_{\infty, s}$, then for any $\rho > 0$ it holds

$$\lim_{n \rightarrow +\infty} \frac{c_n}{n^\rho} = +\infty.$$

3. Selections, games, partition relations and $R_{\infty, s}$

In this section we show that the class of rapidly varying sequences satisfies a selection principle and the corresponding game-theoretical and Ramsey-theoretical conditions.

Let \mathcal{A} and \mathcal{B} be sets whose elements are families of subsets of an infinite set X . Then (see [9]):

$S_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(A_n: n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(b_n: n \in \mathbb{N})$ such that for each n , $b_n \in A_n$ and $\{b_n: n \in \mathbb{N}\}$ is an element of \mathcal{B} .

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(A_n: n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(B_n: n \in \mathbb{N})$ of finite (not necessarily nonempty) sets such that for each n , $B_n \subset A_n$ and $\bigcup_{n \in \mathbb{N}} B_n$ is an element of \mathcal{B} .

The following result shows that the class $R_{\infty, s}$ satisfies a Rothberger-type selection principle well known in the theory of Selection Principles.

Theorem 3.1. *The class $R_{\infty, s}$ satisfies the selection principle $S_1(R_{\infty, s}, R_{\infty, s})$ (and thus $S_{\text{fin}}(R_{\infty, s}, R_{\infty, s})$).*

Proof. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of elements of $R_{\infty, s}$. Suppose that for each n , $s_n = (c_{n,m})_{m \in \mathbb{N}}$. By induction create now a sequence $(a_n)_{n \in \mathbb{N}}$ so that:

- (i) $a_1 = c_{1,k}$ for some $k \in \mathbb{N}$;
- (ii) for $n > 1$, a_n is chosen to be $c_{n,m}$ which is $\geq 2a_{n-1}$.

By Corollary 2.4, such a choice is possible. We claim that $(a_n)_{n \in \mathbb{N}} \in R_{\infty, s}$.

Let $\lambda > 1$. For $n \in \mathbb{N}$ we have

$$\frac{a_{[\lambda n]}}{a_n} = \frac{a_{[\lambda n]}}{a_{[\lambda n]-1}} \cdots \frac{a_{n+1}}{a_n}.$$

On the right-hand side of the previous equality we have $[\lambda n] - n$ factors and also $[\lambda n] - n > (\lambda n - 1) - n = (\lambda - 1)n - 1$. So,

$$\frac{a_{[\lambda n]}}{a_n} \geq 2^{[\lambda n] - n} > 2^{(\lambda - 1)n - 1}, \quad n \in \mathbb{N},$$

and thus

$$\lim_{n \rightarrow +\infty} \frac{a_{[\lambda n]}}{a_n} = +\infty.$$

By Corollary 2.2 one concludes that $(a_n)_{n \in \mathbb{N}} \in R_{\infty, s}$. \square

Notes.

- (a) Theorem 3.1 remains valid if instead of $a_{n+1} \geq 2a_n$, $n \in \mathbb{N}$, we require $a_{n+1} \geq \alpha_n \cdot a_n$, $n \in \mathbb{N}$, where $\liminf_{n \rightarrow +\infty} \alpha_n > 1$.
- (b) Notice that Theorem 3.1 (and its modification described in (a) above) is still valid if instead of the class $R_{\infty, s}$ one takes the class of unbounded sequences of positive real numbers.

There is an infinitely long game associated to the selection principle $S_1(\mathcal{A}, \mathcal{B})$.

The symbol $G_1(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the n th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \dots; A_n, b_n; \dots)$ if $\{b_n: n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if ONE does not have a winning strategy in the game $G_1(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $S_1(\mathcal{A}, \mathcal{B})$ is true. The converse implication needs not be always true, but many properties described by selection principles can be characterized by the corresponding game (see [9]).

Theorem 3.2. *The player TWO has a winning strategy in the game $G_1(R_{\infty, s}, R_{\infty, s})$.*

Proof. Let us define a strategy σ for TWO in the following way. Suppose that in the first round ONE plays $s_1 = (c_{1,m})_{m \in \mathbb{N}}$ from $R_{\infty, s}$. Then TWO responds by choosing $\sigma(s_1) = c_{1,m_1}$, where c_{1,m_1} is any element in s_1 . Let in the second round ONE play $s_2 = (c_{2,m})_{m \in \mathbb{N}}$; TWO (applies Corollary 2.4 and) finds $c_{2,m_2} \in s_2$ such that $c_{2,m_2} \geq 2 \cdot c_{1,m_1}$ and responds by $\sigma(s_1, s_2) = c_{2,m_2}$. If in the n th round ONE has played $s_n = (c_{n,m})_{m \in \mathbb{N}}$, then TWO chooses $c_{n,m_n} \in s_n$ such that $c_{n,m_n} \geq 2^{n-1} \cdot c_{1,m_1}$ and plays $\sigma(s_1, s_2, \dots, s_n) = c_{n,m_n}$. And so on. It is evident, as in the proof of Theorem 3.1, that σ is a winning strategy for TWO. \square

Suppose that \mathcal{A} and \mathcal{B} are as above, $n, k \in \mathbb{N}$, and let for a set A the symbol $[A]^n$ denote the set of all n -element subsets of A . We are going now to show that the class $R_{\infty, s}$ satisfies a combinatorial principle from Ramsey theory known as the *ordinary partition relation*

$$\mathcal{A} \rightarrow (\mathcal{B})_k^n$$

which is the statement:

For each $A \in \mathcal{A}$ and for each function $f: [A]^n \rightarrow \{1, \dots, k\}$ there are a set $B \in \mathcal{B}$ with $B \subset A$ and some $i \in \{1, \dots, k\}$ such that for each $Y \in [B]^n$, $f(Y) = i$.

Note that several selection principles of the form $S_1(\mathcal{A}, \mathcal{B})$ have been characterized by the ordinary partition relation (see [8]).

Theorem 3.3. *The class $R_{\infty,s}$ satisfies the ordinary partition relation*

$$R_{\infty,s} \rightarrow (R_{\infty,s})_k^n, \quad n, k \in \mathbb{N}.$$

Proof. We prove the theorem for $n = k = 2$; by a standard induction argument on n and k , the usual method for proving Ramsey theoretical statements for $n > 2$, $k > 2$ (see, for example, Theorem 1 in [11]), we can prove the general case. Let $s = (c_1, c_2, \dots)$ be a sequence in $R_{\infty,s}$ and let $f : [s]^2 \rightarrow \{1, 2\}$ be a coloring. It is easy to verify that one of the sets $s_1 := \{c_i \in s : f(\{c_1, c_i\}) = 1\}$ and $s_2 := \{c_i \in s : f(\{c_1, c_i\}) = 2\}$ is in $R_{\infty,s}$. Denote by i_1 the element from $\{1, 2\}$ for which s_{i_1} is in $R_{\infty,s}$ and put $q_1 = s_{i_1}$. Inductively define q_n and i_n , $n \geq 2$, such that $q_n := \{c_i \in q_{n-1} : f(\{c_n, c_i\}) = i_n\}$ is a rapidly varying sequence. Apply now $S_1(R_{\infty,s}, R_{\infty,s})$ to the sequence $(q_n)_{n \in \mathbb{N}}$ to choose for each n an element $a_n \in q_n$ such that $a = (a_n)_{n \in \mathbb{N}} \in R_{\infty,s}$. Of course, we may assume that $a_n \neq a_m$ whenever $n \neq m$ and that there exists $i \in \{1, 2\}$ satisfying: for each $a_m \in a$, $i_m = i$. It follows that $f(\{a_l, a_m\}) = i$ for each $\{a_l, a_m\} \in [a]^2$ and so the claim. \square

4. α_i -Properties and rapidly varying sequences

In [10], new selection principles were introduced in the following way; \mathcal{A} and \mathcal{B} are as above.

Definition 4.1. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 1, 2, 3, 4$, denotes the following selection hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is an element $B \in \mathcal{B}$ such that:

- $\alpha_1(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;
- $\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
- $\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
- $\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

Evidently,

$$\alpha_1(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_2(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_3(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B}).$$

It is also clear that

$$S_1(R_{\infty,s}, R_{\infty,s}) \Rightarrow S_{\text{fin}}(R_{\infty,s}, R_{\infty,s}) \Rightarrow \alpha_4(R_{\infty,s}, R_{\infty,s}).$$

We are going to prove that each of properties $\alpha_i(R_{\infty,s}, R_{\infty,s})$, $i = 2, 3, 4$, is equivalent to $S_1(R_{\infty,s}, R_{\infty,s})$.

Theorem 4.2. *The class $R_{\infty,s}$ satisfies the following equivalent statements:*

- (1) $S_1(R_{\infty,s}, R_{\infty,s})$;
- (2) $\alpha_2(R_{\infty,s}, R_{\infty,s})$;
- (3) $\alpha_3(R_{\infty,s}, R_{\infty,s})$;
- (4) $\alpha_4(R_{\infty,s}, R_{\infty,s})$.

Proof. We have to prove only $(1) \Rightarrow (2)$ and $(4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Let $(s_n: n \in \mathbb{N})$ be a sequence of elements of $R_{\infty,s}$. Suppose that for each $n \in \mathbb{N}$, we have $s_n = (x_{n,m})_{m \in \mathbb{N}}$. Choose an increasing sequence $k_1 < k_2 < \dots < k_p < \dots$ of positive integers and for each n and each k_i consider $\sigma(n, k_i) := \{x_{n,m}: m \geq k_i\}$. Then each $\sigma(n, k_i)$, $n, i \in \mathbb{N}$, is a rapidly varying sequence. Apply (1) to the sequence $(\sigma(n, k_i): i \in \mathbb{N}, n \in \mathbb{N})$ of elements of $R_{\infty,s}$ and find a sequence $(y_{n,k_i}: i, n \in \mathbb{N})$ such that for each $(n, i) \in \mathbb{N} \times \mathbb{N}$, $y_{n,k_i} \in \sigma(n, k_i)$ and the sequence $\sigma := (y_{n,k_i})_{n,i \in \mathbb{N}} \in R_{\infty,s}$. It is easy to see that for each $n \in \mathbb{N}$ the set $s_n \cap \sigma$ is infinite, i.e., σ is a selector for the sequence $(s_n: n \in \mathbb{N})$ showing that $\alpha_2(R_{\infty,s}, R_{\infty,s})$ holds.

$(4) \Rightarrow (1)$: Let $(s_n: n \in \mathbb{N})$ be a sequence of elements of $R_{\infty,s}$. Enumerate every s_n bijectively as $s_n = (x_{n,m})_{m \in \mathbb{N}}$. By (4) there is an increasing sequence $n_1 < n_2 < \dots$ in \mathbb{N} and a sequence $s = (x_{n_i, m_i})_{i \in \mathbb{N}} \in R_{\infty,s}$ such that for each $i \in \mathbb{N}$, $x_{n_i, m_i} \in s_{n_i}$. According to Theorem 3.1 one may assume that $x_{n_1, m_1} \geq 2^{n_1-1} \cdot x_{1, m_1}$ for some m_1 , and that for each $i \geq 2$, $x_{n_i, m_i} \geq 2^{n_i-n_{i-1}} \cdot x_{n_{i-1}, m_{i-1}}$. Put $n_0 = 0$. For each $i \geq 0$ and each n with $n_i \leq n < n_{i+1}$ choose $x_{n, m_n} \in s_n$ such that $x_{n, m_n} \geq 2^{n-n_i} \cdot x_{n_i, m_i}$. Then the sequence $(x_{n, m_n}: n \in \mathbb{N})$ witnesses for the sequence $(s_n: n \in \mathbb{N})$ that $S_1(R_{\infty,s}, R_{\infty,s})$ holds. \square

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